

Lectures II-III: Reductive Perturbation Theory and Mode-Jumping

Models: - Free, Forced Duffing
- Nonlinear KFG

Key Idea:

- origin of unphysical secularities
- removal by nonlinear frequency shift

Key Consequences

- nonlinear resonance behavior
- mode jumping.

Important examples:

- free Duffing
- NL Klein-Gordon
- forced Duffing.

2) Nonlinear Oscillators - Conservative

→ Here, concerned with $\left\{ \begin{array}{l} \text{nonlinear} \\ \text{conservative} \end{array} \right\}$ oscillator systems, usually small perturbations about/away from the SHO

Prototype: Duffing's Equation

$$\underbrace{\ddot{x} + \omega_0^2 x}_{\text{SHO}, \omega_0} + \epsilon x^3 = 0$$

\downarrow
 NL, anharmonic term

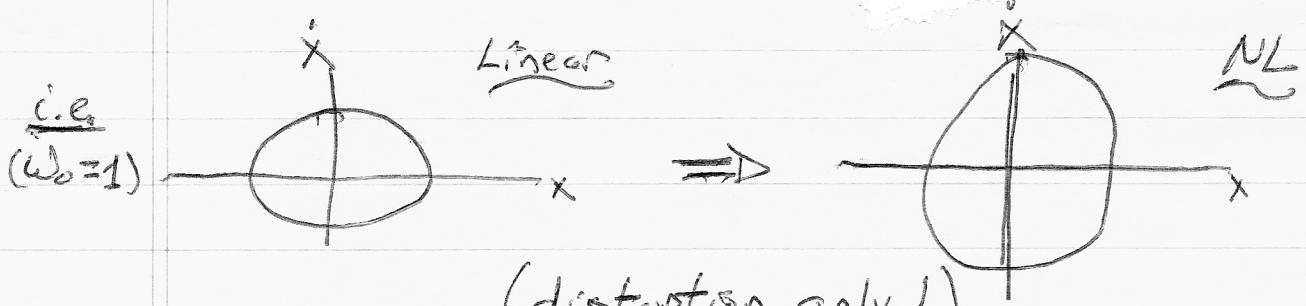
Observe:

$$- H = \frac{1}{2} \dot{x}^2 + \frac{\omega_0^2 x^2}{2} + \epsilon \frac{x^4}{4}$$

$$\text{so } V(x) = \frac{1}{2} \omega_0^2 x^2 + \epsilon \frac{x^4}{4} \quad (V(x)' > 0)$$

- natural question to ask re: $\omega = \omega(\epsilon)$?
i.e. evolution of periodic/quasi-periodic orbits upon perturbation.

Now note: $V(x)$ bounded \Rightarrow phase space contours (from below) trajectory "contained"



(distortion only!)
so orbit must be bounded!

- A clue;

Observe: $\omega^2 = \frac{1}{2} \omega_0^2 \langle x^2 \rangle / \langle x^2 \rangle / 2$
 for SHO, $\langle \rangle = \frac{1}{T} \int_0^T \dots$
 \Rightarrow

might expect, taking $T = 2\pi/\omega_0$

$$\omega^2 = \left(\frac{1}{2} \omega_0^2 \langle x^2 \rangle + \frac{\epsilon}{4} \langle x^4 \rangle \right) / \frac{\langle x^2 \rangle}{2}$$

(solve with SHO 'trial' fctn.)

Now; $\langle x^4 \rangle = \langle a^4 (\cos \omega_0 t)^4 \rangle$
 $= a^4 (3/4)$

$$\omega^2 = \omega_0^2 + \frac{3\epsilon a^2}{4}$$

\rightarrow amplitude dependent frequency! (NL frequency shift)

i.e. $\omega = \omega_0 \rightarrow$

\rightarrow (nearly) correct result.

$\omega = \omega(\omega_0, \epsilon, a)$! \rightarrow frequency becomes amplitude dependent.

Systematics - Computational Procedure

- expand in ϵ ! \leftrightarrow (surprise !!)

$$\ddot{x} + \omega_0^2 x + \epsilon x^3 = 0 \quad ; \quad x(0) = 0$$

$$x = \underbrace{x^{(0)}} + \epsilon x^{(1)} + \dots$$

$\therefore O(\epsilon^0)$: $\ddot{x}^{(0)} + \omega_0^2 x^{(0)} = 0$

start $x^{(0)} = a \sin(\omega_0 t)$
 ϵx^1

$O(\epsilon^1)$: $\ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\cancel{\epsilon} (x^{(0)})^3$

but $(x^{(0)})^3 = a^3 \sin^3 \omega_0 t$

$$= \frac{a^3}{4} (3 \sin \omega_0 t + \sin 3 \omega_0 t)$$

start

$$\Rightarrow \ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\cancel{\frac{\epsilon a^3}{4}} (3 \sin \omega_0 t + \sin 3 \omega_0 t)$$

\downarrow
resonant drive
 \Rightarrow secularity !!

yields $x^{(1)} = \text{homog} + C t \cos \omega_0 t$

\downarrow
is this physical ?

$$\Rightarrow -\omega_0^2 c t \cos \omega_0 t - 2c \omega_0 \sin \omega_0 t + \omega_0^2 c t \cos \omega_0 t = -\frac{\epsilon a^3}{4} (3 \sin \omega_0 t)$$

$$c = \frac{3\epsilon}{8\omega_0} ; \quad \epsilon \sim (\text{freq.})^2, \text{ dimensionally}$$

$$\therefore x = a \sin \omega_0 t + \frac{3}{8} \frac{\epsilon}{\omega_0} t \cos \omega_0 t + \dots$$

secularity \rightarrow $|x|$ diverges linearly in time

Unphysical \Rightarrow recall closed phase space trajectories!

What's going on?? \rightarrow ω shift

Aside: A trivial example! P

Benebat
Crismer
Kevorkian
Cole
Mitropolsky

$$\ddot{x} + (1+\epsilon)^2 x = 0 ; \quad x(0) = 0$$

$$\dot{x}(0) = 1$$

if expand in ϵ ;

$$\ddot{x} + x + 2\epsilon x + \epsilon^2 x = 0 \quad (\text{exactly solvable})$$

$$\epsilon^{(0)} ; \quad \ddot{x}^{(0)} + x^{(0)} = 0 ; \quad x^{(0)} = \sin t$$

$$O(\epsilon); \quad \ddot{x}^{(1)} + x^{(1)} = -2x^{(0)}$$

$$\Rightarrow x^{(1)} = c t \left[\cos t \right] + \text{homog.}$$

$$\Rightarrow c \sin t - c t / \cos t + c t / \cos t = -2 \sin t$$

$$c = 1$$

• secular! ? ?

This is clearly idiotic, since we all know

$$x(t) = \sin[(1+\epsilon)t] \quad \text{trivially solves the problem!}$$

↳ frequency shift

Moral of this story:

- ② - to avoid secular, must allow frequency shift; i.e. here $\omega = 1 \rightarrow \omega = 1 + \epsilon$
gives all a warm, fuzzy.....
- ① - secular results from breakdown of naive expansion in ϵ at long times, observe:

i.e. $x(t) = \sin[(1+\epsilon)t],$

Taylor expansion in $\epsilon \Rightarrow \approx \sin t + \epsilon t \cos t$

↳ secular \rightarrow {artifact of expansion}

→ The Fix:

- admit nonlinear frequency shift

(i.e. method of Poincaré-Linstedt)

Top of Reductive P.T. ice-berg ----

- trick is to:

a) expand x, ω on equal footing

$$x = x^{(0)} + \epsilon x^{(1)} + \dots$$
$$\omega = \omega^{(0)} + \epsilon \omega^{(1)} + \dots$$

$$\ddot{x} + \omega_0^2 (x + \epsilon x^3) = 0$$

and use/choose $\omega^{(1)}$, etc. to cancel secularities (remove)

i.e. "solvability condition" ⇔ secularity removal

i.e. $\frac{d^2 x}{dt^2} + x + \epsilon x^3 = 0$

Scale to ω_0 .
 $\delta = \omega_0$

usual long time behavior

now! $t = \tau (1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots)$
time param.

and $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$

Improved Details

$$* \ddot{X} + \omega_0^2 X + \epsilon X^3 = 0$$

$$\text{Now, } \left\{ \begin{array}{l} X = X^{(1)} + X^{(2)} + X^{(3)} \\ \quad \quad \quad \downarrow \\ \quad \quad \quad \begin{array}{l} O(\epsilon) \\ O(\epsilon^2) \end{array} \end{array} \right. \rightarrow O(\epsilon^3)$$

$$\left\{ \begin{array}{l} \omega = \omega_0 + \omega^{(1)} + \omega^{(2)} + \dots \\ \quad \quad \quad \downarrow \\ \quad \quad \quad O(\epsilon) \text{ (shift)} \end{array} \right.$$

$$\text{and, } X_0 = a \cos \omega t$$

↳ note $\omega \neq \omega_0$
(not ω_0)

Note $\omega \rightarrow \omega_0$
if shift ignored.

Now; re-write:

$$\frac{\omega_0^2}{\omega^2} \ddot{X} + \omega_0^2 X = -\epsilon X^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right) \ddot{X}$$

added
↓

↳ added

added $\frac{\omega_0^2}{\omega^2}$
↳ added

extra terms assure LHS = 0, in lowest order.

⇒

$$\frac{\omega_0^2}{\omega^2} \left(-\omega^3 X^{(1)} + \ddot{X}_2 \right) + \omega_0^2 \left(X^{(1)} + X_2 \right)$$

$$= -\epsilon \left(X^{(1)} + X^{(2)} \right)^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right) \left(-\omega^3 X^{(1)} + \ddot{X}_2 \right)$$

$$\begin{aligned}
 & -\cancel{\omega_0^2} x^{(4)} + \ddot{x}_2 + \cancel{\omega_0^2} x^{(4)} + \omega^2 x_2 \\
 & = -\epsilon x^{(4)3} - (-\omega^2 + \omega_0^2) x^{(4)} \\
 & \quad - \left(1 - \frac{\omega_0^2}{\omega^2} \right) \ddot{x}_2
 \end{aligned}$$

⇒

$$\ddot{x}_2 + \omega_0^2 x_2 = \underbrace{-\epsilon x^{(4)3}}_{O(\epsilon)} + \underbrace{(\omega^2 - \omega_0^2) x^{(4)}}_{O(\epsilon)}$$

$$\begin{aligned}
 & - \left(1 - \frac{\omega_0^2}{\omega^2} \right) \ddot{x}_2 \quad \overset{h.o.}{\sim} \epsilon^2 \\
 & \quad \downarrow \text{shift} \quad \rightarrow \text{d.e. } x_2 \sim O(\epsilon) \\
 & \quad \quad \quad \left(1 - \frac{\omega_0^2}{\omega^2} \right) \sim O(\epsilon)
 \end{aligned}$$

$$\omega^2 = \omega_0^2 + 2\omega_0\omega, \epsilon \quad ; \quad \omega^2 - \omega_0^2 = 2\omega_0\omega, \epsilon$$

$$\begin{aligned}
 x^{(4)3} &= a^3 \cos^3 \omega t = a^3 \left[\frac{1}{2} + \frac{\cos 2\omega t}{2} \right] \cos \omega t \\
 &= a^3 \left[\frac{\cos \omega t}{2} + \frac{1}{2} \left(\frac{1}{2} \right) (\cos 3\omega t + \cos \omega t) \right] \\
 &= a^3 \left[\frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t \right]
 \end{aligned}$$

$$\ddot{X}_2 + \omega_0^2 \dot{X}_2 = -\epsilon \left[a^3 \left(\frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t \right) \right] + 2\omega_0 \omega_1 \epsilon a \cos \omega t$$

Now to dodge the secularity, need cancel all resonant terms on RHS!

⇒

$$\ddot{X}_2 + \omega_0^2 \dot{X}_2 = -\epsilon \left[\cos \omega t \right] \left[\frac{3a^3 - 2\omega_0 \omega_1 a}{4} \right] = \epsilon \frac{a^3}{4} \cos 3\omega t$$

$$\omega_1 = \frac{3}{8} a^2 / \omega_0$$

and

$$\omega = \omega_0 + \epsilon \left(\frac{3}{8} a^2 / \omega_0 \right) + \dots$$

NL frequency shift!

and crank \Rightarrow

$$x^{(2)} = \frac{-1}{2} \left(\frac{6a^3}{16\omega_0^2} \right) \cos 3\omega t.$$

Point:

- need get frequencies correct to avoid unphysical resonances, secularities...
- frequency correction \Rightarrow NL frequency shift.

N.B. : compare:

$$- \text{exact: } \omega_1 = 3/8 a^2 / \omega_0$$

$$- \text{rough: } \omega_1 = 3/4 a^2 / \omega_0$$

Moral: Use frequency shift to eliminate secularities causing term on RHS

So:

$$\frac{d^3}{ds^2} (X_0 + \epsilon X_1 + \epsilon^2 X_2 + \dots) + \left[(X_0 + \epsilon X_1 + \epsilon^2 X_2 + \dots) \right. \\ \left. + \epsilon (X_0 + \epsilon X_1 + \epsilon^2 X_2 + \dots)^3 \right] (1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots)^2 = 0$$

$$O(\epsilon^0): \quad \frac{d^2 X_0}{ds^2} + X_0 = 0$$

$$O(\epsilon^1): \quad \frac{d^2 X_1}{ds^2} + X_1 + X_0^3 + 2\omega_1 X_0 = 0$$

$$O(\epsilon^2): \quad \frac{d^2 X_2}{ds^2} + X_2 = 3X_0^2 X_1 - 2\omega_1 (X_1 + X_0^2) \\ - (\omega_1^2 + 2\omega_2) X_0$$

etc.

Now,

$$O(\epsilon^0): \quad \frac{d^2 X_0}{ds^2} + X_0 = 0$$

$$X_0 = a \cos(s + \phi) \quad \rightarrow \text{phase}$$

$\omega(\epsilon')$:

$$\frac{d^2 x_1}{ds^2} + x_1 = -x_0^3 - 2\omega_1 x_0$$

$$= -a^3 \cos^3(s+\phi) - 2\omega_1 a \cos(s+\phi)$$

etc

Aut

$$\cos^3(s+\phi) = \cos(s+\phi) \left[\frac{1}{2} + \frac{1}{2} \cos[2(s+\phi)] \right]$$

$$= \frac{1}{2} \cos(s+\phi) + \frac{1}{2} \cos(s+\phi) \cos[2(s+\phi)]$$

$$= \frac{\cos(s+\phi)}{2} + \frac{1}{4} \cos[3(s+\phi)] + \frac{1}{4} \cos[(s+\phi)]$$

$$= \frac{3}{4} \cos(s+\phi) + \frac{1}{4} \cos(3(s+\phi))$$

\Rightarrow

①

②

$$\frac{d^2 x_1}{ds^2} + x_1 = -a^3 \left(\frac{3}{4} \cos(s+\phi) + \frac{1}{4} \cos(3(s+\phi)) \right)$$

③

$$- 2\omega_1 a \cos(s+\phi)$$

①, ③ $\sim \cos(st + \phi)$

resonates with RHS \leftrightarrow
will drive secularity.

② $\sim \cos[2(st + \phi)]$

non-secular drive \rightarrow harmless

$$\frac{d^2 x_1}{ds^2} + x_1 = \left[-\frac{3}{4} a^2 - 2\omega_1 \right] a \cos(st + \phi) + \frac{1}{4} \cos(3(st + \phi))$$

so $\left\{ \omega_1 = -\frac{3}{8} a^2 \right.$ removes secularity $\left. \right\}$

$t = s \left(1 - \frac{3}{8} \epsilon a^2 + \dots \right)$

$\omega = \omega_0 \left[1 + \frac{3}{8} \epsilon a^2 + \dots \right]$

and $x_1 = \frac{1}{32} a^3 \cos[3(st + \phi)]$

NL
 frequency
 shift \downarrow

i.e.
 $s = t / \left(1 - \frac{3}{8} \epsilon a^2 \right)$
 $\approx t \left(1 + \frac{3}{8} \epsilon a^2 \right)$

Key Point: \rightarrow By expanding ω in ϵ , method of Poincaré and Lindstedt introduces additional degrees of freedom, so one can remove secularities order-by-order in P.T.

PS of asymptotics

\rightarrow essence of NL oscillator is NL frequency shift, i.e.

$$\omega = \omega_0 + \frac{3\epsilon a^2}{8\omega_0}$$

Forced Anharmonic Oscillator - Mode Jumping

- here, consider next step in development

\Rightarrow forced Duffing's eqn.

$$\ddot{x} + \underbrace{2\lambda \dot{x}}_{\text{friction}} + \omega_0^2 x + \underbrace{\alpha x^2 + \beta x^3}_{\alpha=0, \text{initially}} = \frac{F_0 x}{m}$$

issue: combination $\left\{ \begin{array}{l} \text{resonance} \\ \text{forcing} \\ \text{NL} \end{array} \right.$

→ Frequency shift + recovers guess estimate (almost...)

24th

→ example 2 - Nonlinear Klein Gordon Eqn.
mention

KG eqn:

$$\frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi = 0$$

(Scalar Field)

→ dispersion relation:

$$\omega^2 = c_0^2 k^2 + m^2$$

what physical system does this describe?
⇒ NL pendulum + springs.

(Calc plasma wave)

$$\mathcal{L} = \frac{\dot{\phi}^2}{2c_0^2} - \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 - \frac{m^2 \phi^2}{2}$$

for nonlinearity:

$$U = \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{m^2 \phi^2}{2} + \frac{\alpha \phi^4}{4}$$

$$\Rightarrow \left\{ \frac{\partial^2 \phi}{\partial t^2} - c_0^2 \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi = -\alpha \phi^3 \right.$$

physics?

1/2 wave solution

$$\omega^2 = c_0^2 k^2 + m^2$$

↳ unperf. speed.

look for wave train solutions!

$$\phi = \phi(x-ct) = \phi(\theta)$$

exact speed

sneaky → converts to ODE problem.

So, for NL problem

$$\left[\begin{aligned} (c^2 - c_0^2) \phi'' + m^2 \phi &= -\alpha \phi^3 & \phi' \\ (c^2 - c_0^2) \frac{d^2 \phi}{dx^2} + m^2 \frac{\phi^2}{2} + \alpha \frac{\phi^4}{4} &= \text{const} \end{aligned} \right.$$

∴ expect nonlinearity will produce nonlinear phase velocity shift!

Notation

$$\Rightarrow \left. \begin{aligned} c &= c^{(0)} + \alpha^2 c_2 + \dots \\ \phi &= \alpha \phi_1 + \alpha^3 \phi_3 + \dots \end{aligned} \right\} \begin{array}{l} \text{by correspondence} \\ \text{with Duffing} \\ \text{~~oscillator~~} \end{array}$$

Ans. here $\phi = \phi(x)$ \leftrightarrow wave train solution
∴ only parameter is c

$$\Rightarrow (c^{(0)} + \alpha^2 c_2)^2 - c_0^2 \left[\alpha \phi_1'' + \alpha^3 \phi_3'' \right] + m^2 \left[\alpha \phi_1 + \alpha^3 \phi_3 \right] = -\alpha \left[\alpha \phi_1 + \alpha^3 \phi_3 \right]^3 = 0$$

$$(c^{(0)2} - c_0^2) \phi_1'' + m^2 \phi_1 = 0 \quad O(\alpha)$$

$$(c^{(0)2} - c_0^2) \phi_3'' + m^2 \phi_3 = -2 c^{(0)} c_2 \phi_1''$$

$$\alpha \phi_1^3 \quad O(\alpha^3)$$

now, $O(a^2)$:

$$\phi_1 = \phi_0 \cos k\theta \quad (\text{continued on } a)$$

$$C^{(a)^2} = \omega^2 + \frac{m^2}{k^2}$$

$$\omega^2 = \omega_0^2 - \frac{3}{4} \omega^4$$

$$\omega^2 = \omega_0^2 - \frac{3}{4} \omega^4$$

$O(a^3)$:

$$\begin{aligned} & ((C^{(a)^2} - \omega_0^2) \phi_0'' + m^2 \phi_3) \\ &= -2 C^{(a)^2} C_2 (\cos k\theta)'' \end{aligned}$$

$$- \alpha (\cos k\theta)^3$$

$$= 2 C^{(a)^2} C_2 k^2 \cos k\theta$$

$$- \alpha \cos k\theta \left[\frac{1}{2} + \frac{1}{2} \cos 2\theta k \right]$$

$$= 2 C^{(a)^2} C_2 k^2 \cos k\theta - \frac{\alpha}{2} \cos k\theta$$

$$+ \frac{\alpha}{2} \frac{1}{2} \cos 3k\theta - \frac{\alpha}{2} \frac{1}{2} \cos k\theta$$

$$= 2 C^{(a)^2} C_2 k^2 \cos k\theta - \frac{3}{4} \alpha \cos k\theta$$

$$+ \frac{\alpha}{4} \cos 3k\theta$$

so, to kill secularly:

$$2c^{(0)} c_2 k^2 = \frac{3 \alpha}{4}$$

$$c_2 = \frac{3 \alpha}{8 c^{(0)} k^2}$$

$$\therefore c = c^{(0)} + a^2 c_2$$

$$= \left(c_0^2 + \frac{m^2}{k^2} \right)^{1/2} \left[1 + \frac{3 \alpha a^2}{8 k^2 c^{(0)2}} \right]$$

$$c = \left(c_0^2 + \frac{m^2}{k^2} \right)^{1/2} \left[1 + \frac{3 a^2 \alpha}{8 (c_0^2 k^2 + m^2)} \right]$$

Nonlinear speed change /
shift

Recall, for linear forced SHO; (see 31)

$$a^2 = F^2 / 4m^2 \omega_0^2 (\epsilon^2 + \lambda^2) \quad \text{is } \begin{cases} \text{amplitude} \\ \text{equation} \end{cases}$$

$$\epsilon = \omega - \omega_{res} ; \quad \omega_{res} = \omega_0 \quad (\text{trivial!})$$

Then for NL system: $\omega_{res} = \omega_0 + K a^2$
 (near primary, linear resonance) $K = \frac{3\epsilon\beta}{8\omega_0}$ \rightarrow NL shift \rightarrow shift

$$a^2 = F^2 / 4m^2 \omega_0^2 ((\omega - \omega_0 - K a^2)^2 + \lambda^2)$$

$$a^2 [(\epsilon - K a^2)^2 + \lambda^2] = F^2 / 4m^2 \omega_0^2$$

$\left\{ \begin{array}{l} a(\epsilon) \text{ relation} \\ \Rightarrow \text{cubic equation for } a^2 \end{array} \right.$

\Rightarrow cubic equation for a^2 !! (3 roots \rightarrow which?)
 Contrast above

Observe:

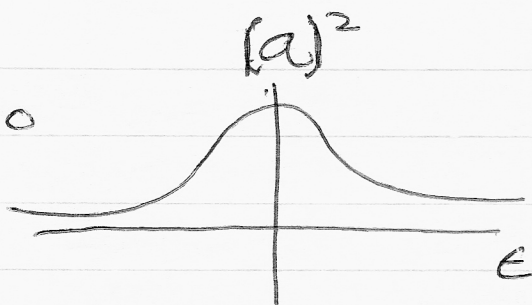
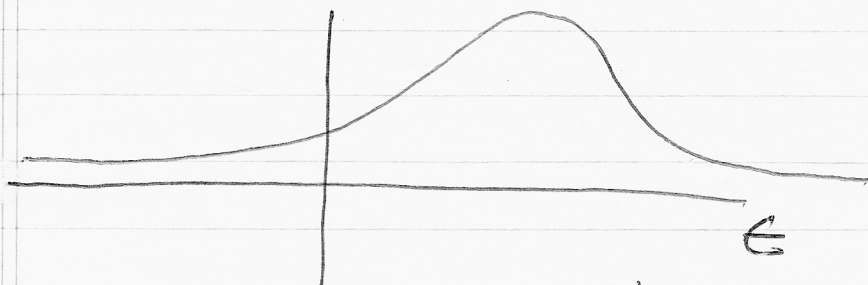
- addition of NL \Rightarrow NL ω -shift \Rightarrow
non-trivial amplitude equation

- for $\omega = \omega_{res}$ ($\epsilon = K a^2$), $a^2 = F^2 / 4m^2 \omega_0^2 - \lambda^2$
 peak unchanged

but

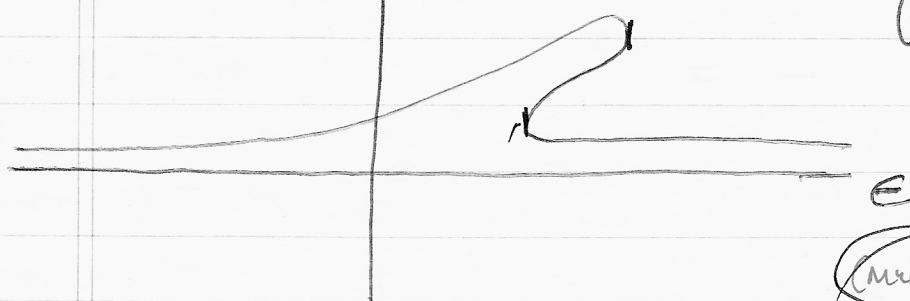
asymmetry induced in resonance curve!

i.e.

 $R=0$ i.e. Lorentzian
(symmetric) $R \neq 0$ \Rightarrow a^2 

(asymmetric)

|||

 a^2 sharpness $\sim 1/\lambda \Rightarrow$ NL more
impt!

(double valued!)

(mult)

Can the resonance curve be double-valued?
 \rightarrow yes! \Rightarrow when $da^2/d\epsilon \rightarrow \infty$.

Now, amplitude equation: $\ddot{a}(\epsilon)$

$$a^2 [(\epsilon - ka^2)^2 + \lambda^2] = f^2 / 4m^2\omega_0^2$$

$$\Rightarrow a^2 [\epsilon^2 - 2k\epsilon a^2 + (ka^2)^2 + \lambda^2] = f^2 / 4m^2\omega_0^2$$

Can re-write amplitude equation as:

$$F(a^2, \epsilon) = a^2 [\epsilon^2 - 2\epsilon(Ka^2) + (Ka^2)^2 + \lambda^2] - \frac{F^2}{4m^2\omega_0^2} = 0$$

So, for $\frac{da^2}{d\epsilon}$ on curve (defn.)

$$dF = 0 = (\partial F / \partial a^2) da^2 + (\partial F / \partial \epsilon) d\epsilon$$

$$\Rightarrow \frac{da^2}{d\epsilon} = - \frac{(\partial F / \partial \epsilon)}{(\partial F / \partial a^2)}$$

∴ $\partial F / \partial a^2 = 0$ for double valuedness (infinite slope) 2 roots → though $\partial F / \partial a^2 = 0$ is coalescence pt. of two onset blow up.

$$F = K^2(a^2)^3 - 2\epsilon K(a^2)^2 + a^2(\epsilon^2 + \lambda^2) - \text{const.}$$

$$\partial F / \partial a^2 = 3K^2(a^2)^2 - 4\epsilon K a^2 + (\epsilon^2 + \lambda^2) = 0$$

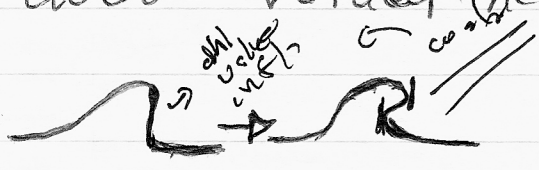
if $x = Ka^2$

$$\partial F / \partial a^2 = 0 = 3x^2 - 4\epsilon x + (\epsilon^2 + \lambda^2)$$

$$\Rightarrow x = \frac{4\epsilon}{6} \pm \frac{1}{6} (16\epsilon^2 - 12(\epsilon^2 + \lambda^2))^{1/2}$$

$$= \frac{2}{3}\epsilon \pm \frac{1}{3} (\epsilon^2 - 3\lambda^2)^{1/2}$$

$$\underline{\text{so}} \quad ka^2 = \frac{2}{3} \epsilon \pm \frac{1}{3} (\epsilon^2 - 3\lambda^2)^{1/2}$$

" → double valuedness at inflection pt.

 } 2 - root coalescence
mult

i.e. when $\epsilon^2 = 3\lambda^2$, $\Rightarrow ka^2 = 2\epsilon/3$

in terms external force magnitude ($a(\epsilon)$ relation)

$$F^2 = 4m^2 \omega_0^2 a^2 \left[\epsilon^2 - 2\epsilon ka^2 + (ka^2)^2 + \lambda^2 \right]$$

$$\begin{cases} ka^2 = \frac{2}{3} \epsilon = \frac{2\sqrt{3}}{3} \lambda & (\text{pk}) \quad \& \\ \epsilon = \sqrt{3} \lambda & (\text{pk}) \quad \& \end{cases}$$

and plugging into $F^2 \Rightarrow$

$$F_{\text{crit}}^2 = 32m^2 \omega_0^2 \lambda^3 / 3\sqrt{3} |K|$$

i.e. $F > F_{\text{crit}}$
 required
 for inflection

→ for peak value of amplitude

$$\frac{\partial F}{\partial \epsilon} = 0 \quad (\Leftrightarrow \frac{da^2}{d\epsilon} = 0)$$

$$\Rightarrow 2\epsilon - 2ka^2 = 0 \Rightarrow \epsilon = ka^2$$

$$\Rightarrow \omega - \omega_0 = ka^2, \text{ usual!}$$

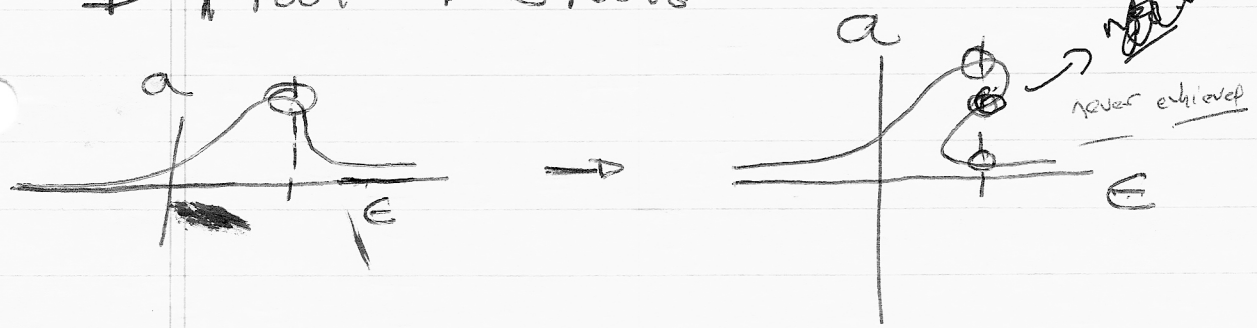
i.e. peak is at resonance (here $\omega = \omega_0 + R a^2$), as usual.

→ What's Going On?

- for $f^2 > f_{crit}^2 = 32 m^2 \omega_0^2 \lambda^3 \sqrt{3} |R|$

⇒ bifurcation occurs → bifurcation

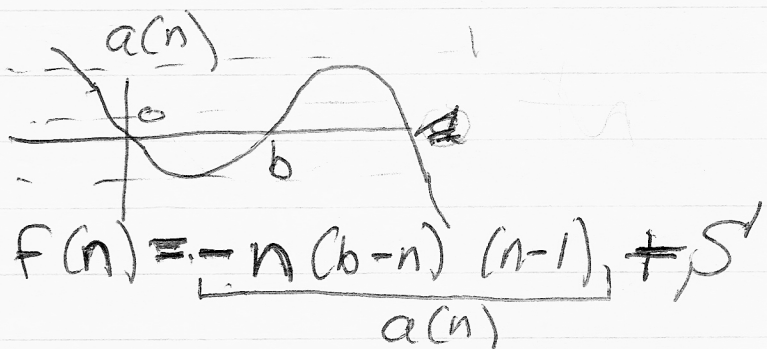
⇒ 1 root → 3 roots



- of 3 roots ⇒ 2 stable
1 unstable

i.e. demonstration

$$\frac{\partial n}{\partial t} = f(n)$$



$$\frac{\partial n}{\partial t} = 0 \Rightarrow S = +n(b-n)(n-1)$$

control ⇒ 3 or 1 roots, depending on S'

but $n = n_{sol} + \delta n$

$\frac{\partial n}{\partial t} = \delta n f'(n_{sol})$

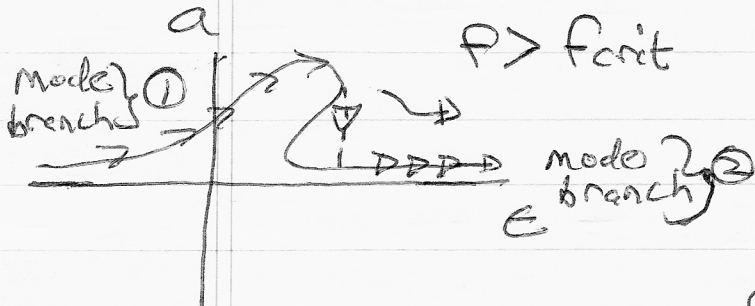
$f' > 0 \rightarrow$ instability
 $f' < 0 \rightarrow$ stability

i.e. $f' > 0 \rightarrow$ unstable root (1)

$f' < 0 \rightarrow$ stable roots (2)

mode branching

- 'bifurcation' occurs \rightarrow jump between 2 stable branches (bifurcation \leftrightarrow inflection criterion)



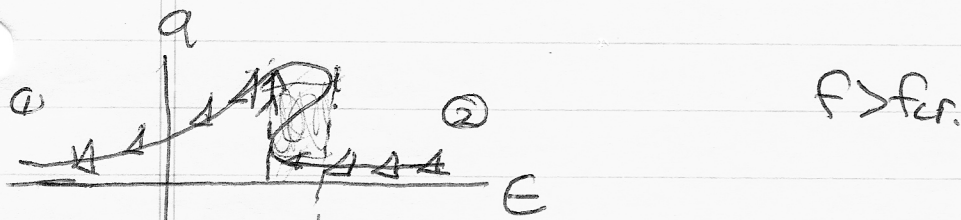
$a(f, \epsilon) \rightarrow$ surface catastrophe

$\epsilon > \epsilon_{crit} \Rightarrow$ jump from {branch mode 1} to {branch mode 2}

- system exhibits "hysteresis"

i.e.

consider reversal of evolution from (2) \rightarrow (1), i.e.

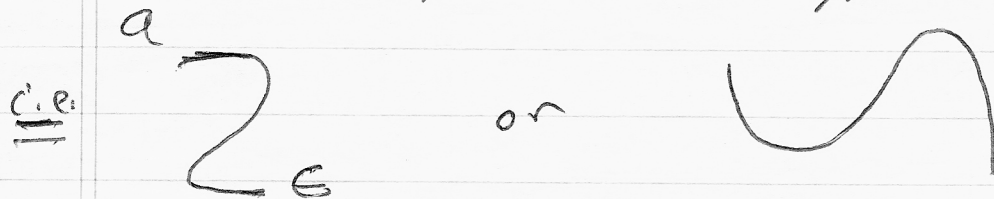


E_{crit} for forward transition
 E_{crit} for back transition

c.e.

$E_{fwd} > E_{back}$
 system tends to 'hang' in mode ②

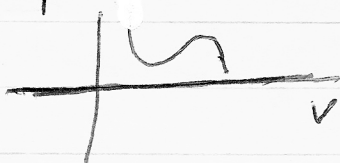
→ driven Duffing oscillator is classic example of "S-curve" type bifurcation



S-curve ⇒ bi-stable system with unstable root in between, yielding transitions or mode-jumping

⇒ akin phase transition
 ①, ② ⇔ 2 phases

S curve ⇔ p v curve



mode jumping ⇔ first order transition

→ Now,

- have examined impact of NL on resonance phenomena, at primary/linear resonance

- but, is this the whole story?

⇒ NL-induced resonance phenomena? → $\left\{ \begin{array}{l} \text{new} \\ \text{resonance} \\ \text{physics} \end{array} \right.$

Now, re-insert αx^2 term!

$$\ddot{x} + 2\lambda \dot{x} + \omega_0^2 x + \alpha x^2 + \beta x^3 = \frac{f_{\text{ext}}}{m}$$

↑
quadratic
NL

here

$$\omega_{\text{res}} = \omega_0 + K a^2$$

$$K = \left(\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3} \right)$$

why $O(\alpha^2)$?
→ $x^{(2)} x^{(1)}$ beat

↓
shift contribution,
due quadratic → derive

Why α^2 ?

- observe $-\alpha (x^{(1)})^2 \rightarrow \frac{1}{2} \cos(2\omega_0 t)$ > non-resonant
drives to $O(a^2)$

⇒ lowest secular contribution from αx^2
nonlinearity is in $x^{(1)}$ equation
here power a , not ϵ !